Speed-up through entanglement – many-body effects in neutrino processes

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Abstract

We study a system containing many particles of identical kinematics with a zero range interaction that scatters one from the other, and with the possible exchange of an attribute. Taking an initial condition in which the attribute is asymmetrically distributed in the regions of momentum space occupied by the particles, we study the rate at which it becomes uniformly distributed, through collisions. We find, in some circumstances, a rate that is much faster than that which would be estimated from cross-sections. This behavior is attributable in some general sense to N-particle entanglement. We suggest applications to neutrino physics, where the attribute is neutrino flavor.

1 Introduction

We consider a system consisting of stable or quite long-lived particles (i.e. stable for the time range that in which we shall be interested), confined in a box for simplicity, and interacting occasionally with one another. Taking an initial state which is, at least with respect to some of its attributes, not in statistical equilibrium, we can discuss time scales for evolution of the gross features of the system. In general these scales are determined by cross-sections. But an exception to this assertion about scales and cross-sections can be found, among other places, in the transformation of one state of a particle into another

through evolution that can occur in isolation from other particles, such as in neutrino oscillations or the precession of spins in external magnetic fields. In this case it may not be to the point to discuss the instantaneous "rates" at which the system changes, since we are watching coherent development of an attribute of the system with time behavior generically like $\sin \omega t$.

In the present note we demonstrate some circumstances under which we do not have such slow single-particle "precessions" but in which there is a long term coherent process arising from the short range interactions of particles, which can be randomly distributed in momentum space. This behavior can arise when the particles have another attribute that can be traded in the course of an interaction. Both to have a concrete framework and with a view to a possible application, we shall consider a system of neutrinos. In order to eliminate extraneous effects, we consider an example in which there are two flavors of massless neutrinos, designated ν_e and ν_τ, and there is no neutrino flavor mixing in the Hamiltonian. There is an initial distribution of the neutrinos specified, one in which there is some systematic flavor asymmetry, such as neutrinos of one flavor being predominantly of higher energy than those of the other flavor, or with different angular distributions for the two species. The question we pose is: "At what later time would the distributions would become more or less equal through neutral current interactions?" The conventional answer, as mentioned above, is that this time is determined by a scattering rate proportional to the weak cross-section times the density of scatterers. Equations in the literature do indeed predict shorter time scales for effects arising from neutrino-neutrino interactions in the case in which there is flavor oscillation built into the Hamiltonian [1]. The effects studied in these papers are analogous to the "forward scattering", or "index of refraction" terms familiar in the study of the passage of neutrinos through matter that contains electrons, in that they have an inverse time scale that is proportional to $G_F \rho$ where ρ is the neutrino number-density of the medium and G_F is the Fermi constant. This is in contrast to cross-section effects, which are of order $G_F^2 \rho \omega^2$, where ω is the energy of the scattering particles. These studies, however, predict that such effects are strictly absent in the absence of flavor mixing in the Hamiltonian, if one one begins, as we do, with a state that is diagonal in the flavor space [2]. The essential difference between our approach and that of previous work is that we retain much of the full complexity of the multi-body physics involved, rather than assuming we may describe our ensemble of particles with a single body density matrix. [1]

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[1] Some of the subtleties involved in neutrino-neutrino forward scattering have recently been re-examined by A. Friedland and C. Lunardini [3]. In particular, they raise the point that there may be circumstances in which a single body description is inadequate, as is certainly the case for the effects we study here. However, we do not see how to obtain the results of the present paper using the perturbative approach of these authors.

2 Model

The basic process that we consider is simply

$$\nu_e(\mathbf{p}) + \nu_\tau(\mathbf{q}) \to \nu_e(\mathbf{q}) + \nu_\tau(\mathbf{p}), \tag{1}$$

where the momenta $\mathbf{p}$ and $\mathbf{q}$ are drawn from the initial distributions. These processes will tend to reduce the correlations of flavor and momentum that we assume are present in the initial state. The question is: "At what rate?" The part of the Hamiltonian that will provide our effects operates only in the subspace of the initial momentum states. We define annihilation operators a_i for a ν_e of momentum p_i and b_i for a ν_τ of momentum p_i, where i runs from unity to the number of single particle momenta.[2] Given an initial state of N_1 momenta occupied by ν_e and N_2 momenta occupied by ν_τ, we take the effective Hamiltonian that implements the full set of reactions in Eq. (1) to be,

$$H_I^{(eff)} = \frac{1}{2} \frac{\sqrt{2} G_F}{V} \sum_{i \neq j}^{N_1 + N_2} f_{ij} \, [a_j^\dagger a_i b_i^\dagger b_j + \, a_i^\dagger a_j b_j^\dagger b_i] \tag{2}$$

where V is the volume of the system and the weight function f_{ij} is of order unity. The sum extends over all of the $(N_1 + N_2)$ momentum states of the system that are initially occupied by either flavor of neutrino. We have omitted the $i = j$ terms in the Hamiltonian since the corresponding processes, $\nu_e(p_i) + \nu_\tau(p_j) \to \nu_e(p_i) + \nu_\tau(p_j)$, do not contribute to our effects. Including such terms, that is to say, a diagonal contribution to the Hamiltonian, alters the wavefunction of the system only by an irrelevant overall phase. Since the energies of the basic set of unperturbed states are exactly the same, the time evolution of the system will be entirely determined by this interaction Hamiltonian.

In a realistic problem the form of f_{ij} will depend on the circumstances of the application. For example, if we had started with the complete form of the neutral current, neutrino-neutrino interactions, then the matrix elements of the Dirac matrices in the V,A structure dictate a factor of $[1 - \cos(\theta_{p_i,p_j})]$, where the angle is that between the two momentum vectors labeling the states. For simplicity, we have taken $f_{ij} = 1 \; \forall i, j$ for some of our analytic estimates. In the numerical calculations we have taken a distribution of values in the

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[2] Not necessarily equal to the number of particles, since in the initial state a ν_e and a ν_τ could both have the same momentum.

range $0.5 - 1$, given by

$$f_{ij} = 0.5 + \frac{0.5}{N_1 + N_2} |i - j| .$$

(3)

3   N+N

In our first example we shall take a set of $2N$ momentum states to be occupied half by ν_e's and half by ν_τ's. For the initial state we take the first (bottom) N states to be filled by ν_τ's and the last (top) N states to be filled by ν_e's. We refer to this state as $|\Psi_0\rangle$. Explicitly,

$$|\Psi_0\rangle = |\nu_e(p_1) \dots \nu_e(p_i) \dots \nu_e(p_N)\nu_\tau(p_{N+1}) \dots \nu_\tau(p_j) \dots \nu_\tau(p_{2N})\rangle .$$

(4)

The interactions given in Eq.(1) exchange energy and momentum between particles of the two different species. An example of such a process is the interchange of our initial state with

$$|\nu_e(p_1) \dots \nu_\tau(p_i) \dots \nu_e(p_N)\nu_\tau(p_{N+1}) \dots \nu_e(p_j) \dots \nu_\tau(p_{2N})\rangle .$$

(5)

The total set of states of the complete system that we have to deal with, $|\Psi_\alpha\rangle$, are the $n_s = (2N)!/(N!)^2$ distinct states in which the flavor indices in the initial state are permuted within the defined subset indexed by p_i. We shall adopt an ordering of these states such that in each of the first $n_s/2$ states in the list, the top state in the single particle list is occupied by ν_e. Having taken all of the N of the ν_e's on the top, in the initial state, we wish to estimate the time for "equilibration", in the sense of the ν_e 's being more or less distributed with 50% in the bottom set of states.

In the n_s dimensional space spanned by our basis states, $|\Psi_\alpha\rangle$, the effective Hamiltonian matrix as determined from Eq.(2) has N^2 off-diagonal elements in each row (or column); we denote this matrix by $\mathbf{M}$. Each of the $n_s N^2$ off-diagonal elements of $\mathbf{M}$ is one of the 2N(2N-1)/2 values of the function f_{ij} of Eq. (2).

Taking $f_{ij} = 1$, we have,

$$\langle \Psi_\alpha | \mathbf{M}^2 | \Psi_\alpha \rangle = \frac{2N^2 G_F^2}{V^2}$$

(6)

for each one of the basis states $|\Psi_\alpha\rangle$. The square root of the average squared

energy of the n_s eigenstates is thus given by

$$E_{av} = \left[n_s^{-1} Tr[\mathbf{M}^2] \right]^{1/2} = \frac{1}{2} \sqrt{2} G_F \rho \tag{7}$$

where the trace operates in our space of n_s states, and we have substituted the total $(\nu_e + \nu_\tau)$ number-density, ρ, for $2N/V$. We denote the corresponding set of eigenstates of M by $|\Psi_E\rangle$, where the E's stand for the n_s eigenvalues. The state of the system at time T, which we denote by $|\Psi(T)\rangle$ is given by

$$|\Psi(T)\rangle = \sum_E^{n_s} |\Psi_E\rangle\langle\Psi_E|\Psi_0\rangle e^{-iET}, \tag{8}$$

From this we can compute, for example, the probability P (persistence) that a particular one of the top $n_s/2$ states, which was occupied at time T=0 by a ν_e, is occupied at time T by a ν_τ. For example, with the $|\Psi_0\rangle$ and the ordering of the n_s states as described above, we ask for the probability of continuing to have a ν_e in the very top state, obtaining

$$P = \sum_{\alpha=1}^{n_s/2} |\langle\Psi_\alpha|\Psi(T)\rangle|^2, \tag{9}$$

where the sum is over all the $n_s/2$ states that have a ν_e on top.

Of course at the moment we have said nothing about the coefficients $\langle\Psi_\alpha|\Psi_E\rangle$ that enter in Eq. (9) and Eq. (8). We anticipate that an eigenvector of a matrix such as M typically projects significantly onto many of the basis states $|\Psi_\alpha\rangle$. Our numerical tests sustain this conclusion, which we will not quantify further in this note. Then the time in which P becomes significantly less than unity on the average, i.e. the mixing time, is determined by the magnitude of a typical eigenvalue. From Eq. (7) we already found that the root mean square of an eigenvalue is of order $\rho\,G_F$. If a macroscopic fraction of the eigenvalues, that is, a number λn_s of them, are of this order,[3] then the effective mixing time should be of order $(\rho\,G_F)^{-1}$, in view of the relations Eq.(8) and Eq.(9). This is the same as the typical time scale for what the "index of refraction" or "forward-scattering" effects would be in the case of in which we replaced the neutrino density by an electron density of the same magnitude, a time scale much shorter than any effect of nonforward scatterings (which scale as G_F^{-2}.)

The distribution of the eigenvalues can be determined analytically in the case in which $f_{ij} = 1$. Consider a system in which the number of ν_e and ν_τ is given

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[3] Of course, if essentially all of the strength were concentrated in a number of eigenvalues which did not grow as n_s, then we would not obtain macroscopic effects.

by N_1 and N_2 respectively. We shall assume, without loss of generality, that $N_1 \leq N_2$. The distribution of eigenvalues have a pattern, from which we can observe that there are $N_1 + 1$ distinct eigenvalues

$$E_i = \frac{\sqrt{2}G_F}{V}[(N_1 - i)(N_2 - i) - i], \qquad i = 0, 1 \ldots N_1, \tag{10}$$

with degeneracies given by

$$D_i = \frac{(N_1 + N_2)!}{i!(N_1 + N_2 - i)!} - \frac{(N_1 + N_2)!}{(i-1)!(N_1 + N_2 - (i-1))!}. \tag{11}$$

In Fig. 1 we plot the distribution of these eigenvalues for the present case of N each of ν_e and ν_τ, in the limit of large N. On the same figure we show the distributions obtained numerically, in the case that f_{ij} is given by Eq. (3), for the cases $2N = 12$ and $2N = 14$. Since the matrix M is traceless, the eigenvalues sum to zero. One can see from Fig. (1) that the bulk of the eigenvalues (say, $> 95\%$) lie in the range

$$-\frac{1}{2} \leq \frac{E_i}{2N} \frac{V}{\sqrt{2}G_F} \leq +1, \tag{12}$$

and thus the typical energy difference between a pair of eigenstates will be of order $(\sqrt{2}G_F/V)(2N) \equiv \sqrt{2}G_F\rho$. The qualitative features of the eigenvalue distribution, are not affected by taking a distribution of f_{ij}'s.
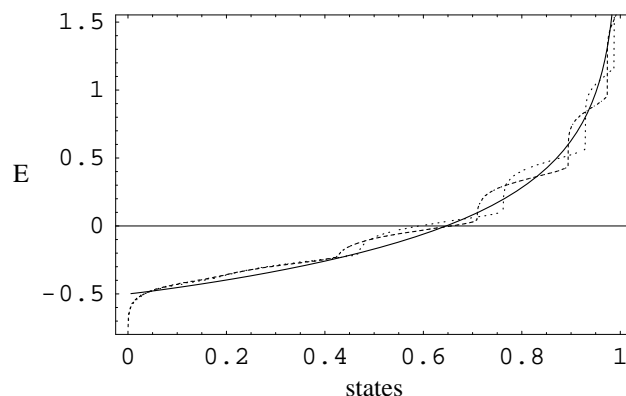

Fig. 1. The distribution of energy eigenvalues for a system of N ν_e's and N ν_τ's, where the energy is in units of $\sqrt{2}G_F\rho$. A point on the curve represents the fraction of states with energy below E. The heavy curve is the limiting result for large $2N$ for the case $f_{ij} = 1$. The heavier and lighter dashed curves are the results for $2N=14$ and $2N=12$ respectively, with f_{ij} given by Eq. (3), but multiplied by a factor of $4/3$ so that the average coupling strengths in the comparisons are the same.

We also make a limited computational check on our heuristic estimates of the effects on rates, by directly solving the Schrodinger equation, beginning

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with the above initial state, Eq. (4), and determining the probability that a particular one of the ν_e's in one of the upper states has been replaced at a later time by a ν_τ. In the case $2N = 14$ we have to solve $n_s = 14!/(7!)^2 = 3432$ coupled linear differential equations, which is our computational limit. The results of these calculations are shown in Fig. 2 for a case in which the coupling function f_{ij} is given by Eq. (3). We see that the curves show very similar behavior, and except for the smallest case with 6 particles, the times elapsed to the first minimum are very close to each other. The results appear to sustain the analysis, given above, for a mixing rate that is independent of particle number for fixed density. Note, however, that the locations for, say the first peaks in the respective curves move to greater time as N is increased, albeit at a slower and slower rate. One cannot absolutely conclude from these data that there is a limiting point for large N and fixed density. Provisionally, taking the mixing time to be that in which the curves cross in Fig. 2, we find $T_{\mathrm{mix}} \approx (\sqrt{2} G_F \rho)^{-1}$ in the limit of an infinite system. Unfortunately, a perturbation theoretic approach is valid only for small times (smaller and smaller times as the particle number is increased) and cannot shed any light on the outcome.
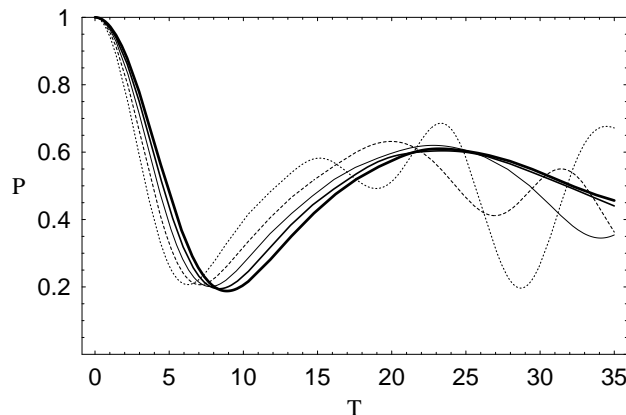

Fig. 2. The mixing parameter, or single level persistence probability, P, as defined in text, as a function of time, for a system of N ν_e and N ν_τ. The curves correspond to the number of particles: $2N=6$, 8, 10, 12 14, where the heaviest curves are for highest N. The unit of time is $(\sqrt{2} G_F \rho)^{-1}$.

4 1+(2N-1)

Next we look at a problem that is so closely related to the above one that one might (erroneously) guess that the behavior is similar. We start with the same Hamiltonian Eq. (2), and set of $2N$ participating states but now take an initial condition with only the top state filled with a ν_e and all the remaining $(2N-1)$ single-particle states filled with ν_τ's. The effective subspace of system states that are connected together is now $n_s = 2N$ dimensional, and

is described by the location of the single ν_e. But now, in the analogue of Eq. (6), for the case of $f_{ij} = 1$ we find that the average squared eigenvalue is $(2N)^{-1}\mathrm{Tr}\mathbf{M}^2 = (2N-1)(\sqrt{2}G_F/V)^2$, in contrast to the value, N^2, obtained in the (N, N) case. Furthermore, there is a single state, with eigenvalue $(2N-1)$, while the remaining $(2N-1)$ states all have eigenvalue -1. Denoting the state with large eigenvalue by Ψ_S, we note that $|\langle\Psi_S|\Psi_0\rangle|^2 = N^{-1}$ so that the effects on P of mixing with this state are of order N^{-1}; and since there are no energy differences among the remaining $N-1$ eigenstates, the total effect (at the order G_F level) will vanish in the limit of large N. We have confirmed numerically that the introduction of a scatter into the coupling constants, using Eq. (3), does not change these conclusions qualitatively. The results for persistence versus time in this case are shown in Fig. 3, for the case of several values of N. They clearly show the N^{-1} behavior in the short and intermediate time regions.
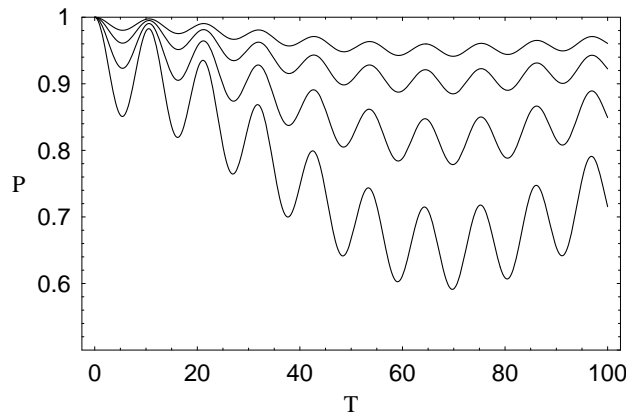

Fig. 3. The same as Fig.2 except for the case of an initial system state with only one particle state occupied by a ν_e and the others occupied by ν_τ's, with P the probability that this the ν_e is in its initial state. We compare the cases $2N$=30, 60, 120, 240 where the slower evolution corresponds to the higher number of particles.

5   Discussion

Thus far in our discussion, we have used "ν_e" and "ν_τ" merely to label our states, and we have given no hint as to how the above behaviors could make a difference to observable results in the neutrino world. In fact there are two situations that have been discussed in the literature in which neutrino-neutrino scattering is thought to play a role, and in which the size of the parameters is exactly such as to make effects with our time scale $(\sqrt{2}G_F\rho)^{-1}$ of possible importance, and indeed as important as the effects noted in the literature. The first is in early universe scenarios in which one has assumed some degree of neutrino degeneracy [4]. The second is in the region just above the neutrino-sphere in the supernova process, where differences in the energy spectra for

different neutrino flavors play a key role in whether or not there is an efficient R process for nucleosynthesis [5]. Another example where neutrino-neutrino forward scattering is relevant, is the flavor conversion of high energy neutrinos from astrophysical sources as they propagate through the relic neutrino background [6]. We emphasize again that up to this point we have not included the neutrino oscillation physics that drives the results in these works. We intend to address the combination of the two themes in a subsequent publication. But we can comment on what we believe to be the insufficiency of the previous works on the subject. What we have computed above, the probability that a specific state, say p_i, that was originally occupied by a ν_e (in the initial state for the complete system $|\Psi_0\rangle$) is still occupied by a ν_e at time t, has a simple expression in terms of the Heisenberg picture operator, $a_i(t)$,

$$P = \langle\Psi_0|a_i^\dagger(t)a_i(t)|\Psi_0\rangle, \tag{13}$$

where the Heisenberg operators are chosen to coincide with the Schrodinger operators that were introduced in Eq. (2) at $t = 0$. Now the operator in Eq. (13) is a neutrino density operator for the particular state p_i. If we write an equation of motion for this operator, by taking the commutator with the Hamiltonian Eq. (2), the right hand side is a quartic in the operators $a^\dagger, a, b^\dagger, b$, and there is a sum over one index j. When we encounter, say, a term like

$$\sum_j a_j^\dagger(t)b_j(t)b_i^\dagger(t)a_i(t) \tag{14}$$

we would <u>like</u> to be able to replace $\sum_j (a_j^\dagger b_j)$ by a density matrix element, off-diagonal in the flavor space, which represents the entire average state of the medium. Then the multiplying operator $b_i^\dagger a_i$ would be a corresponding operator but for the single mode i. Indeed, this is exactly the assumption that yields the non-linear terms in the equations for the density matrices derived in Ref. [2] and used in Ref. [4,5].

As noted above, with our assumed flavor-diagonal initial conditions, and in the absence of neutrino-mixing in the Hamiltonian, such terms do not create any effect. Our effects are exactly due to the fact that the replacement of the four-operator product by a product of two expectation values, as sketched above, is not justified. Even if we were to assume that a kind of factorized ansatz would pick up the leading terms, which we believe is highly unlikely, we would want add the results of pairing, say, $a_j^\dagger a_i$ in Eq. (14). That is to say, we would be driven to consider density matrices that are off-diagonal in momentum space, as well as in flavor space.

To summarize, we have evidence for a new, macroscopic quantum effect that could change the outcome of calculations in which ν-ν scattering matters. The caveats that must be added are:

(i) Accepting the earlier conclusions for the model defined by the Hamiltonian Eq. (2), we must go back and ask whether we have defined the correct problem. All those modes that we left out, when we truncated our Hamiltonian to momentum states that were initially occupied by one species or the other, can they make a difference? These additional modes would be populated via non-forward scattering interactions, which enter the problem at the level of cross-section ($\times$ density), that is, on time scales of order $(G_F^2 \rho \omega^2)^{-1}$. Our expectation is that they will not affect our results on the much shorter time scales we consider (of order $(G_F \rho)^{-1}$.

(ii) Our calculations pertain directly to plane waves in a volume V. Ordinary neutrino oscillation theory has come under fire repeatedly from authors that suspect that preparation-of-state considerations bring the idealized, plane-wave picture into question. These criticisms have been successfully answered more than once [8] for the case of single-particle oscillation, but they could surface again in our present context.

At the present time, we are cautious about claiming that our results will be important to the neutrino physics either in the early universe or in the supernova. These are, however, the two most obvious applications in which the neutrino number density is high enough for neutrino-neutrino scattering to be important. We further note that the flavor-energy correlation is critical to understanding the physics just outside the supernova neutrinosphere. Of course, we would need to include neutrino mass and mixing in the model in order to address a realistic situation. Superficially, we can note that the energies attributable to mass effects are small compared to the inverse time associated with our processes under the conditions that prevail just outside the neutrinosphere of a supernova. But more analysis is required before reaching a conclusion in that context. A negative feature of our work is that in contrast to being able to use the quite simple equations for the density matrix posited by Ref. [2], we appear to be doomed to treating the full complexity of the many-body physics that arises in these systems.

6 An example with a finite number of discrete states

We demystify the physics of much of the above, to some degree, by thinking of a fairly large but finite set of discrete quantum subsystems with interactions of comparable strength between each pair. In addition to being an interesting exercise this also is relevant to applications in other arenas than the world of neutrinos. We consider a set of N spins associated with one set of fixed site labels, say red, and another set of N spins associated with a second set of labels, say blue. The interaction exchanges all pairs of spins with comparable strength, or order g, for each pair. Let us now start with all reds up and all

blues down, and consider the short time evolution of this state. The probability that the spin on a particular red site becomes down in a very short time t is of order $h = g^2 N t^2$. We might think that this perturbation calculation stays more or less valid until such a time that h is, say, 0.1. But if we go back through the same mechanics that we used for our continuum problem, we see that is not be the case; the system will mix in a time T such that $gNT = 1$ [4], at which time the above perturbative estimate of the amount of mixing is only $h = 1/N$.

Next consider the case in which the first red spin is up, and every other red and blue spin is down. In complete correspondence with the (1, 2N-1) neutrino case, we find that the above perturbative estimate of the mixing time, for the first red spin to become appreciably blue, is correct. The mixing time for this single state is longer than that of the first example by a factor of N, for large N.

What makes these cases so different from each other? In both cases the spin at the first red site interacts with each of the N blue spins. But in the second example, the blue spins themselves interact only with that one red site, at the turn-on time, while in the first example every blue spin is being affected by every red spin from the beginning. The difference lies not in the number of interactions that the red spin sees, but in the entanglements of the states that it interacts with. Since in the first case we get faster evolution than we would have expected from the perturbation estimate we classify the effect as a "speed-up". And the reason is clearly the multiparticle entanglement within the system. For completeness, we should note that even in the second example, the blue sites, with which the spin on the distinguished red site interacts, do develop mutual entanglement, but with a much slower initial rate than in the first example. In effect, they see each other only through their mutual coupling to the distinguished red site.

7   Conclusion

We have examined a "speed-up" of evolution through entanglement, both in a discrete system of spins and in a neutrino model with a continuum of states. Though the practical meaning of "speed-up" is slightly different in the two cases, the formal source of the effect is the same.

The reader might ask the question: "Are the authors using the word 'entan-

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[4] Note that in this case we do not redefine the coupling constant with increasing N; there is no volume or density involved in the models. Thus the eigenvalues grow as N.

glement' in some precise mathematical sense, or are they just using it to say the system gets complicated?" We contend that it is the former, in the sense that the state of the system cannot, in general, be factorized into a product of single particle states. Note, however, that the quantification of multiparticle entanglement is not a concept that that has been precisely defined in the literature. Definitions of two-state entanglement, however, have received much attention. The reader can consult Ref. [7] for a demonstration of how in the simplest relevant ordinary Schrodinger example, with two particles in a double well, entanglement in a precise mathematical sense is spontaneously generated if the two particles interact with each other.

Returning to the neutrino example, the first order effects that mixed the states had an inverse time-scale $\sqrt{2}G_F\rho$; this is the "speeded-up" rate. The usual inverse time-scale for (non-forward) scattering effects is of order $G_F^2\rho\omega^2$ where ω is of the order of the particle energies and much, much slower, than the "speeded-up" rate when ω is of order an MeV. Comparing again with the calculation with only one ν_e in a sea of ν_τ's, where there is no speed-up, the difference is attributable to the degree of entanglement.

The inclusion of entanglement requires abandoning the single-particle description of the system, to include the full complexity of the many-body physics involved. Single-body descriptions do predict significant neutrino-neutrino forward scattering effects [1,2], which, however, are absent for the flavor diagonal initial states we consider here. The results of our many-body calculation may thus have interesting consequences in situations where ν-ν scattering is important.

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